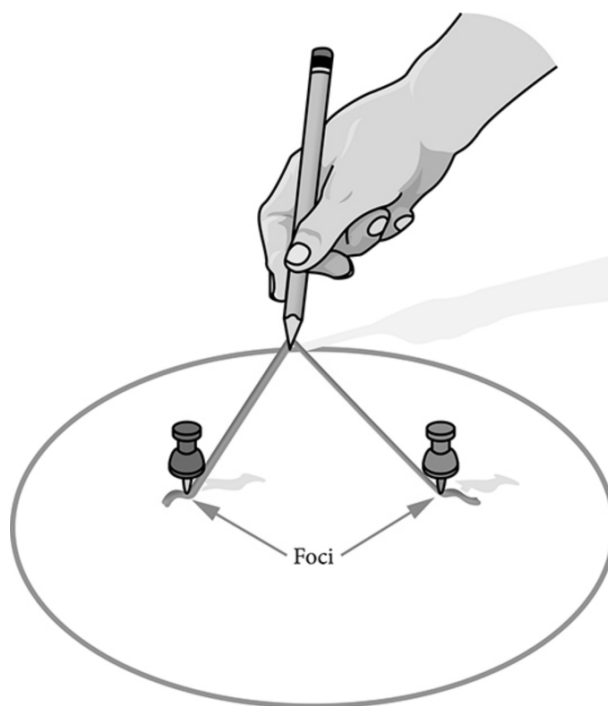


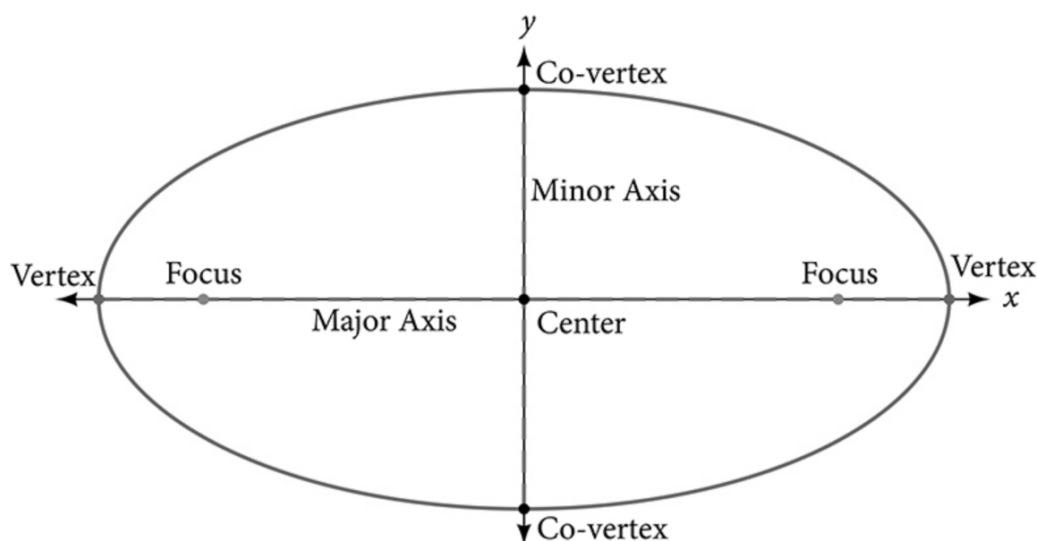
Ellipse

Figure 2

An **ellipse** is the set of all points  $(x, y)$  in a plane such that the sum of their distance from two fixed points is a constant. Each fixed point is called a **focus** (plural **foci**).



Every ellipse has two axes of symmetry. The longer axis is called the **major axis**, and the shorter is called the **minor axis**. Each endpoint of the major axis is called a **vertex** of the ellipse, and each endpoint of the minor axis is a **co-vertex** of the ellipse. The **center of the ellipse** is midpoint of both major and minor axis. The axes are perpendicular at the center. The foci always lie on the major axis, the sum of the distance from the foci to any point on the ellipse (the constant sum) is greater than the distance between the foci.



### Deriving the Equation of an Ellipse Centered at the Origin

To derive the equation of an ellipse centered at the origin, we begin with the foci  $(-c, 0)$  and  $(c, 0)$ . The ellipse is the set of all points  $(x, y)$  such that the sum of the distances from  $(x, y)$  to the foci is constant, as shown in [Figure 5](#).

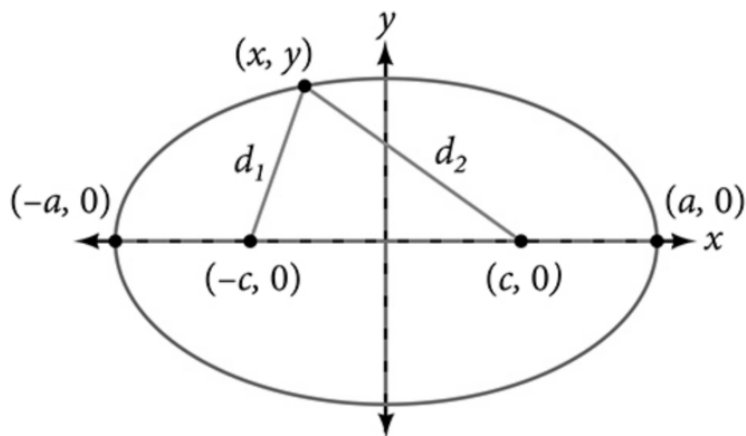


Figure 5

If  $(a, 0)$  is a vertex of the ellipse, the distance from  $(-c, 0)$  to  $(a, 0)$  is  $a - (-c) = a + c$ . The distance from  $(c, 0)$  to  $(a, 0)$  is  $a - c$ . The sum of the distances from the foci to the vertex is

$$(a + c) + (a - c) = 2a$$

If  $(x, y)$  is a point on the ellipse, then we can define the following variables:

$d_1$  = the distance from  $(-c, 0)$  to  $(x, y)$

$d_2$  = the distance from  $(c, 0)$  to  $(x, y)$

By the definition of an ellipse,  $d_1 + d_2$  is constant for any point  $(x, y)$  on the ellipse. We know that the sum of these distances is  $2a$  for the vertex  $(a, 0)$ . It follows that  $d_1 + d_2 = 2a$  for any point on the ellipse. We will begin the derivation by applying the distance formula. The rest of the derivation is algebraic.

$$d_1 + d_2 = \sqrt{(x - (-c))^2 + (y - 0)^2} + \sqrt{(x - c)^2 + (y - 0)^2} = 2a$$

Distance formula

$$\sqrt{(x + c)^2 + y^2} + \sqrt{(x - c)^2 + y^2} = 2a$$

Simplify expressions.

$$\sqrt{(x + c)^2 + y^2} = 2a - \sqrt{(x - c)^2 + y^2}$$

Move radical to opposite side.

$$(x + c)^2 + y^2 = [2a - \sqrt{(x - c)^2 + y^2}]^2$$

Square both sides.

$$x^2 + 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x - c)^2 + y^2} + (x - c)^2 + y^2$$

Expand the squares.

$$x^2 + 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x - c)^2 + y^2} + x^2 - 2cx + c^2 + y^2$$

Expand remaining squares.

$$2cx = 4a^2 - 4a\sqrt{(x - c)^2 + y^2} - 2cx$$

Combine like terms.

$$4cx - 4a^2 = -4a\sqrt{(x - c)^2 + y^2}$$

Isolate the radical.

$$cx - a^2 = -a\sqrt{(x - c)^2 + y^2}$$

Divide by 4.

$$[cx - a^2]^2 = a^2 [\sqrt{(x - c)^2 + y^2}]^2$$

Square both sides.

$$c^2x^2 - 2a^2cx + a^4 = a^2 (x^2 - 2cx + c^2 + y^2)$$

Expand the squares.

$$c^2x^2 - 2a^2cx + a^4 = a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2$$

Distribute  $a^2$ .

$$a^2x^2 - c^2x^2 + a^2y^2 = a^4 - a^2c^2$$

Rewrite.

$$x^2 (a^2 - c^2) + a^2y^2 = a^2 (a^2 - c^2)$$

Factor common terms.

$$x^2b^2 + a^2y^2 = a^2b^2$$

Set  $b^2 = a^2 - c^2$ .

$$\frac{x^2b^2}{a^2b^2} + \frac{a^2y^2}{a^2b^2} = \frac{a^2b^2}{a^2b^2}$$

Divide both sides by  $a^2b^2$ .

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Simplify.

This equation defines an ellipse centered at the origin. If  $a > b$ , then the major axis is horizontal, and if  $b > a$ , then the major axis is vertical.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

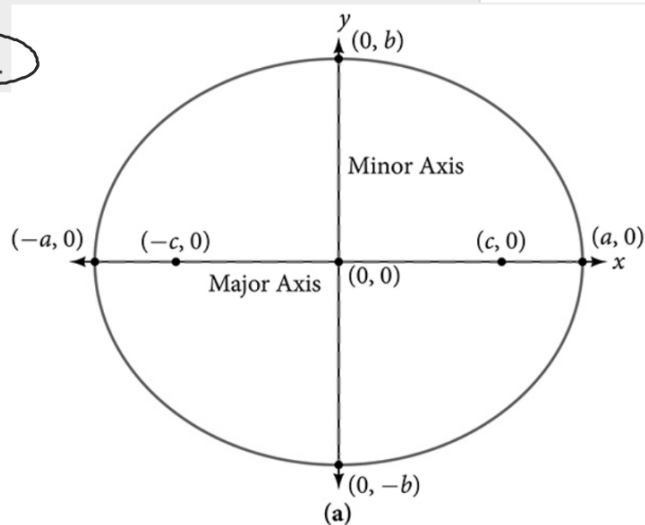
## STANDARD FORMS OF THE EQUATION OF AN ELLIPSE WITH CENTER (0,0)

The standard form of the equation of an ellipse with center  $(0, 0)$  and major axis on the  $x$ -axis is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where

- $a > b$
- the length of the major axis is  $2a$
- the coordinates of the vertices are  $(\pm a, 0)$
- the length of the minor axis is  $2b$
- the coordinates of the co-vertices are  $(0, \pm b)$
- the coordinates of the foci are  $(\pm c, 0)$  where  $c^2 = a^2 - b^2$ .

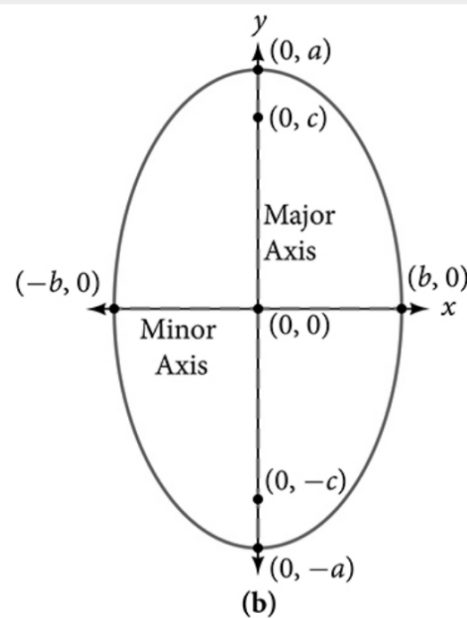


The standard form of the equation of an ellipse with center  $(0, 0)$  and major axis on the  $y$ -axis is

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

where

- $a > b$
- the length of the major axis is  $2a$
- the coordinates of the vertices are  $(0, \pm a)$
- the length of the minor axis is  $2b$
- the coordinates of the co-vertices are  $(\pm b, 0)$
- the coordinates of the foci are  $(0, \pm c)$ , where  $c^2 = a^2 - b^2$ .

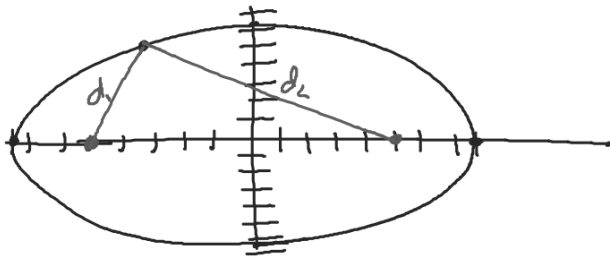


## HOW TO

**Given the vertices and foci of an ellipse centered at the origin, write its equation in standard form.**

1. Determine whether the major axis lies on the  $x$ - or  $y$ -axis.
  - a. If the given coordinates of the vertices and foci have the form  $(\pm a, 0)$  and  $(\pm c, 0)$  respectively, then the major axis is the  $x$ -axis. Use the standard form  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ .
  - b. If the given coordinates of the vertices and foci have the form  $(0, \pm a)$  and  $(\pm c, 0)$ , respectively, then the major axis is the  $y$ -axis. Use the standard form  $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ .
2. Use the equation  $c^2 = a^2 - b^2$ , along with the given coordinates of the vertices and foci, to solve for  $b^2$ .
3. Substitute the values for  $a^2$  and  $b^2$  into the standard form of the equation determined in Step 1.

What is the standard form equation of the ellipse that has vertices  $(\pm 8, 0)$  and foci  $(\pm 5, 0)$



$$\frac{x^2}{64} + \frac{y^2}{39} = 1$$

$$a = 8$$

$$b = \sqrt{39} \approx 6.2$$

$$c = 5$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$c^2 = a^2 - b^2$$

$$5^2 = 8^2 - b^2$$

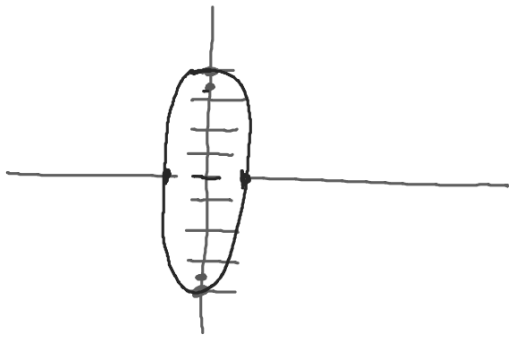
$$25 = 64 - b^2$$

$$-64 \quad -64$$

$$-39 = -b^2$$

$$b^2 = 39$$

What is the standard form equation of the ellipse that has vertices  $(0, \pm 4)$  and foci  $(0, \pm\sqrt{15})$



$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

$$\frac{x^2}{1} + \frac{y^2}{16} = 1$$

$$a = 4$$

$$b = ?$$

$$c = \sqrt{15}$$

$$c^2 = a^2 - b^2$$

$$15 = 16 - b^2$$

$$b^2 = 1$$

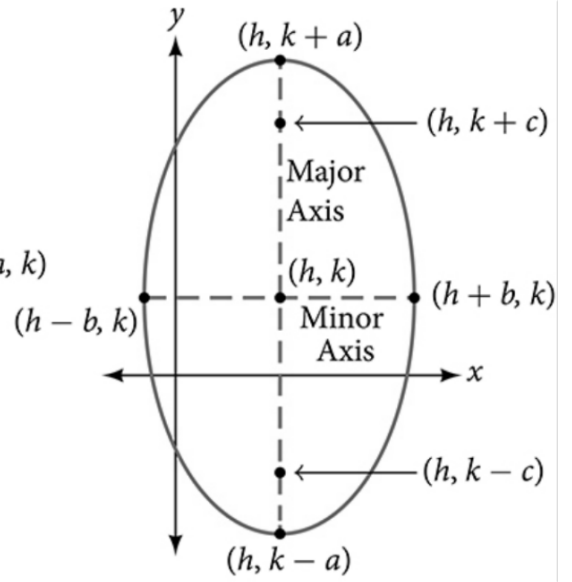
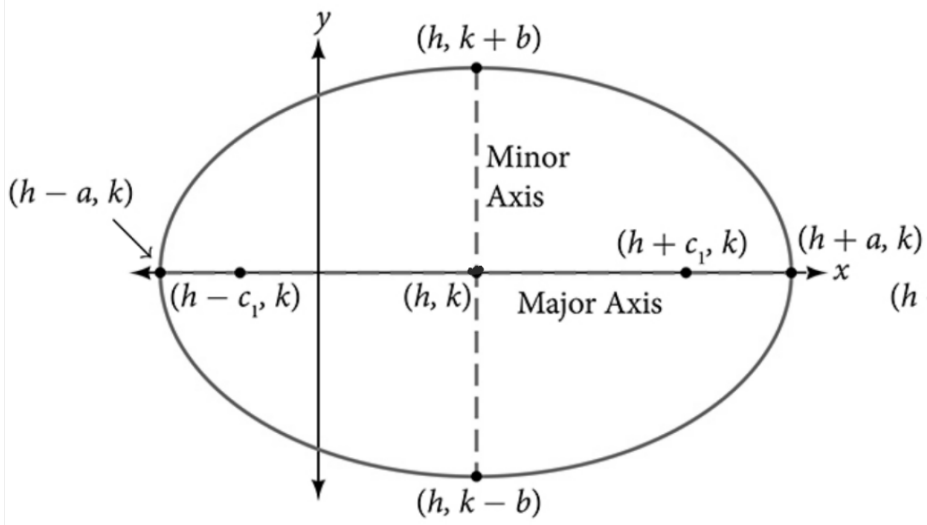
The standard form of the equation of an ellipse with center  $(h, k)$  and major axis parallel to the  $x$ -axis is

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

The standard form of the equation of an ellipse with center  $(h, k)$  and major axis parallel to the  $y$ -axis is

$$\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1$$

where



## HOW TO

**Given the vertices and foci of an ellipse not centered at the origin, write its equation in standard form.**

1. Determine whether the major axis is parallel to the  $x$ - or  $y$ -axis.
  - a. If the  $y$ -coordinates of the given vertices and foci are the same, then the major axis is parallel to the  $x$ -axis. Use the standard form  $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ .
  - b. If the  $x$ -coordinates of the given vertices and foci are the same, then the major axis is parallel to the  $y$ -axis. Use the standard form  $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$ .
2. Identify the center of the ellipse  $(h, k)$  using the midpoint formula and the given coordinates for the vertices.
3. Find  $a^2$  by solving for the length of the major axis,  $2a$ , which is the distance between the given vertices.
4. Find  $c^2$  using  $h$  and  $k$ , found in Step 2, along with the given coordinates for the foci.
5. Solve for  $b^2$  using the equation  $c^2 = a^2 - b^2$ .
6. Substitute the values for  $h$ ,  $k$ ,  $a^2$ , and  $b^2$  into the standard form of the equation determined in Step 1.

### Writing the Equation of an Ellipse Centered at a Point Other Than the Origin

What is the standard form equation of the ellipse that has vertices  $(-2, -8)$  and  $(-2, 2)$

and foci  $(-2, -7)$  and  $(-2, 1)$ ?

Center  $(-2, -3)$

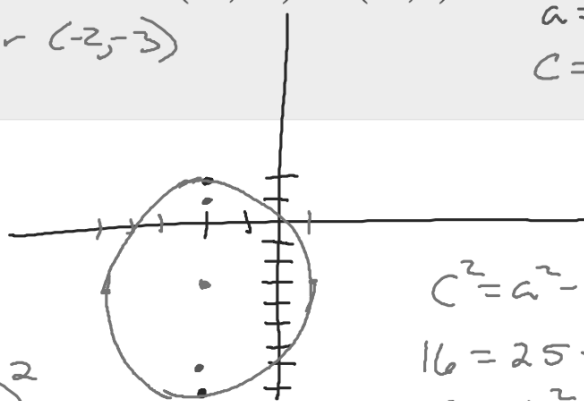
$$\begin{aligned}2a &= 10 \\a &= 5 \\c &= 4\end{aligned}$$

$$\left( \frac{-2 + -2}{2}, \frac{-8 + 2}{2} \right)$$

$$\begin{pmatrix} -2 & -3 \\ h & k \end{pmatrix}$$

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

$$\frac{(x+2)^2}{9} + \frac{(y+3)^2}{25} = 1$$



$$\begin{aligned}c^2 &= a^2 - b^2 \\16 &= 25 - b^2 \\-9 &= -b^2 \\b^2 &= 9 \\b &= 3\end{aligned}$$